The critical indices of the Quark-Gluon Bags with Surface Tension Model with tricritical endpoint

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The critical indices α', β, γ' and δ of the Quark Gluon Bags with Surface Tension Model with the tricritical endpoint are calculated as functions of the usual parameters of this model and two newly introduces parameters (indices). They are compared with the critical exponents of other models. It is shown that for the newly introduced indices $\chi = 0$ and $\xi^T \leq 1$ there is a branch of solutions for which the critical exponents of the present model and the statistical multifragmentation model coincide, otherwise these models belong to different universality classes. It is shown that for realistic values of the parameter $\varkappa$ the critical exponents α', β, γ' and δ of simple liquids and 3-dimensional Ising model can be only described by the branch of solutions in which all indices except for α' correspond to their values within the statistical multifragmentation model. The scaling relations for the found critical exponents are verified and it is demonstrated that for the standard definition of the index α' the Fisher and Griffiths scaling inequalities are not fulfilled for some values of the model parameters, whereas the Liberman scaling inequality is always obeyed. Although it is shown that the specially defined index α'_s recovers the scaling relations, another possibility, an existence of the non-Fisher universality classes, is also discussed.

Keywords: deconfinement phase transition, tricritical endpoint, critical indices

I. INTRODUCTION

Investigation of the properties of strongly interacting matter equation of state has become a focal point of modern nuclear physics of high energies. The low energy scan programs performed nowadays at CERN SPS and BNL RHIC are aimed at the discovery of the (tri)critical endpoint of the quantum chromodynamics (QCD) phase diagram. Despite many theoretical efforts neither an exact location nor the properties of the QCD (tri)critical endpoint are well known [1]. Therefore, the thorough theoretical investigation of the QCD endpoint properties are required in order to clarify whether this endpoint is critical or tricritical.

Here I would like to study the critical exponents of the QCD matter tricritical endpoint (triCEP). Since such a task, unfortunately, cannot be solved within the QCD itself and even the possibilities of the lattice QCD are nowadays very limited in this respect, then, unavoidably, one has to use some models of the triCEP. The most popular models of this kind are the quark-meson model [3, 4] and the extended Nambu–Jona-Lazinio model [5]. Their triCEPs are generated by the intersection of the deconfinement phase transition (PT) line and the chiral symmetry restoration transition line. Since both kind of these models are the mean-field ones there is no reason to expect that their critical exponents could differ from that ones of the Van der Waals model equation of state [6]. Although very recently there appeared an interesting attempt to go beyond the mean-field approximation in the quark-meson model [7], the resulting temperature of the triCEP is about 32 MeV only, which is too low and unrealistic compared to the lattice QCD predictions [2]. Therefore, to study the properties of non-classical triCEPs one has to investigate the non-mean-field models.

Recently there appeared a comprehensive analysis [8] of the QCD phase diagram with the triCEP within an extended version of the gas of bags model [9], which, however, is based on a physically inadequate assumption. Thus, in [8] it is assumed that the Fisher topological exponent τ (denoted as α in [8]) of the Hagedorn mass spectrum should depend on the baryonic chemical potential. However, there is no reason to believe that this exponent characterizing the fractal properties of the surface of large bags should depend on chemical potential. Moreover, the results on the size distribution of clusters obtained for the 2- and 3-dimensional Ising model by the Complement method [10] clearly show that even far away from the critical point the Fisher exponent τ has the same value as at the endpoint.

Therefore, instead of model [8] here I consider another extension of the gas of bags model [9] which is known as the Quark Gluon Bags with Surface Tension Model (QGBSTM) [11, 12]. This exactly solvable model employes the same mechanism of the triCEP generation which is typical for the liquid-gas PT and which is used in the Fisher droplet model (FDM) [13] and in the statistical multifragmentation model (SMM) [14, 15]: the endpoint of the 1-st order PT appears due to vanishing of the surface tension coefficient at this point which leads to the indistinguishability between the liquid and gas phases. However, the surface tension coefficient in the QGBSTM has the region of negative values, which is a principally different feature of this model compared to the FDM, SMM and all other statistical models of the liquid-gas PT. Note that just this feature provides an existence of the cross-over at small values of baryonic

chemical potential μ in the QGBSTM with triCEP [11] and with CEP [15], and also it generates an additional PT in the model with triCEP at large values of μ. Therefore, it is very important and interesting to study the critical indices of such a novel statistical model as the QGBSTM, to determine its class of universality and to examine how the latter is related to that ones of the FDM and SMM.

The work is organized as follows. A brief description of the QGBSTM with the triCEP is given in Section II. In Section III the QGBSTM is analyzed in details and its critical exponents are calculated. Section IV is devoted to the analysis of scaling relations between the found critical exponents. Conclusions are given in Section V.

II. QUARK GLUON BAGS WITH SURFACE TENSION MODEL

An exact solution of the QGBSTM was found in [11]. The relevant degrees of freedom in this model are the quark gluon plasma (QGP) bags and hadrons. The attraction between them is accounted like in the original statistical bootstrap model [17] via many sorts of the constituents, while the repulsion between them is introduced a la Van der Waals equation of state [9, 11]. An essential element of the QGBSTM is the T and μ dependence of its surface tension coefficient $T\Sigma(T_\Sigma, \mu)$ (here $\Sigma(T_\Sigma, \mu)$ is the reduced surface tension coefficient). Let us denote the nil line of the reduced surface tension coefficient as $T_\Sigma(\mu)$, i.e. $\Sigma(T_\Sigma, \mu) = 0$. Note that for a given μ the surface tension is negative (positive) for T above (below) $T_\Sigma(\mu)$ line.

In the grand canonical ensemble the pressure of QGP and hadronic phase are, respectively, given by

$$p_Q(T, \mu) \;=\; T s_Q(T, \mu), \tag{1}$$

$$p_H(T, \mu) \;=\; T\left[F_H(s_H, T, \mu) + u(T, \mu) I_\tau(\Delta s, \Sigma)\right], \tag{2}$$

where the following notations

$$F_H(s_H, T, \mu) \;=\; \sum_{j=1}^{n} g_j e^{\frac{b_j \mu}{T} - v_j s_H} \phi(T, m_j), \tag{3}$$

$$I_\tau(\Delta s, \Sigma) \;=\; \int\limits_{V_0}^{\infty} \frac{dv}{v^\tau} e^{-\Delta s v - \Sigma v^\varkappa}, \tag{4}$$

are used. Here $s_H \equiv \frac{p_H(T,\mu)}{T}$, $\Delta s \equiv s_H(T, \mu) - s_Q(T, \mu)$ and the particle density of a hadron of mass m_j, baryonic charge b_j, eigenvolume v_j and degeneracy g_j is denoted as $\phi_j(T, m_j) \equiv \frac{1}{2\pi^2} \int\limits_0^\infty p^2 dp\, e^{-\frac{(p^2\, +\, m_j^2)^{1/2}}{T}}$. The QGBSTM is solved for a wide range of the functions $u(T, \mu)$ and $s_Q(T, \mu)$ [11]. These functions are the parameters of the present model and here, as in [11], it is assumed that the functions $u(T, \mu)$ and $s_Q(T, \mu)$ and their first and second derivatives with respect to T and μ are finite everywhere at the $T-\mu$ plane. As it was shown in [11], the 1-st order deconfinement PT exists for the Fisher exponent $1 < \tau \leq 2$. In the continuous part of the spectrum of bags defined by (4) the surface of a QGP bag of volume v is parameterized by the term $v^\varkappa$. Usually one chooses $\varkappa = \frac{2}{3}$ in 3-dimensional case (or $\varkappa = \frac{d-1}{d}$ for the dimension d), but in what follows it is regarded as free parameter of the range $0 < \varkappa < 1$.

The system pressure, that corresponds to a dominant phase, is given by the largest value between p_Q and p_H for each set of T and μ. As usual, the deconfinement PT occurs when pressure of QGP gets equal to that one of the hadron gas. Suppose that the necessary conditions for the deconfinement PT outlined in [11] are satisfied and its transition temperature is given by the function $T_c(\mu)$ for $\mu \geq \mu_{cep}$. The necessary condition of the triCEP occurrence is that at the nil line of the surface tension coefficient there exists the surface tension induced PT of 2-nd (or higher order) [11] for $\mu \geq \mu_{cep}$ and $T_c(\mu) \leq T_\Sigma(\mu)$ for these μ values. The both of these inequalities become equalities only at triCEP. Moreover, at triCEP the phase coexistence curve $T_c(\mu)$ is a tangent (not intersecting!) line to the nil line of the surface tension coefficient $T_\Sigma(\mu)$. For $\mu < \mu_{cep}$ the deconfinement PT degenerates into a cross-over since in this region $\Sigma(\mu) < 0$ [11] and, hence, the system pressure is defined by a solution of Eq. (2). A schematic picture of the phase diagram in $\mu - T$ plane is shown in Fig. 1.

An actual parameterization of the reduced surface tension coefficient $\Sigma(T, \mu)$ is taken from [11] ($\zeta = const$):

$$\Sigma(T, \mu) = \frac{\sigma_0}{T} \cdot \left| \frac{T_\Sigma(\mu) - T}{T_\Sigma(\mu)} \right|^\zeta \mathrm{sign}\left(T_\Sigma(\mu) - T\right)\,, \tag{5}$$

but here it is written in a more general form which allows one to perform the calculations for the integer and real values of power ζ. In what follows the coefficient σ_0 is assumed to be a positive constant, i.e. $\sigma_0 = \text{const} > 0$, but

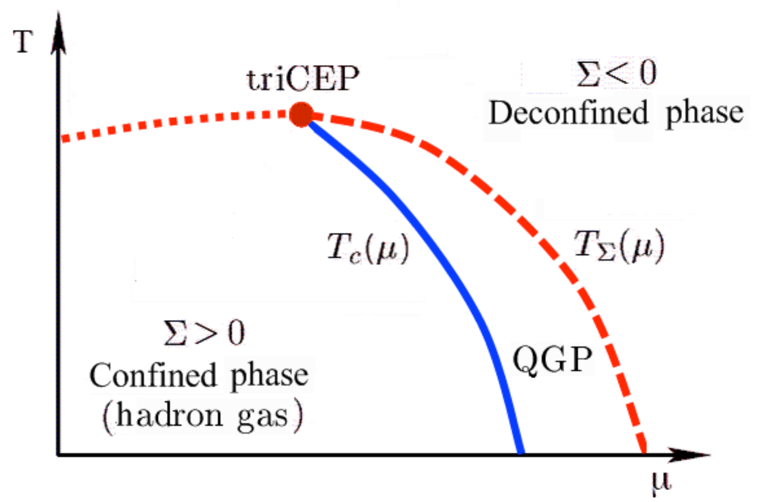

FIG. 1: [Color online] A schematic phase diagram in the plane of baryonic chemical potential μ and temperature T. The dashed curve indicates the nil line of the surface tension coefficient $T_\Sigma(\mu)$, below (above) which the surface tension coefficient is positive (negative). The deconfinement PT line $T_c(\mu)$ is shown by the full curve for $\mu > \mu_{cep}$ whereas the surface tension induced PT is depicted by the long dashed curve for $\mu > \mu_{cep}$. A cross-over (shown by the short dashed curve) takes place along the line $T_\Sigma(\mu)$ for $\mu < \mu_{cep}$. The cross-over and PT regions are separated by triCEP (filled circle). At this point the curves $T_c(\mu)$ and $T_\Sigma(\mu)$ are tangent to each other.

it is easy to show that the obtained results hold, if $\sigma_0 > 0$ is a smooth function of T and μ. Here it is appropriate to say a few words about the negative values of Σ. Note that the existence of regions with $\Sigma < 0$ is a distinctive feature of QGBSTM compared to other models. There is nothing wrong or unphysical with the negative values of surface tension coefficient, since $T\Sigma\, v^{\varkappa}$ is the surface free energy of the bag of mean volume v and, hence, as any free energy, it contains the energy part e_{surf} and the entropy part s_{surf} multiplied by temperature T [13]. Therefore, at low temperatures the energy part dominates and surface free energy is positive, whereas at high temperatures the number of bag configurations with large surface drastically increases and it exceeds the Boltzmann suppression and, hence, the surface free energy becomes negative since $s_{surf} > \frac{e_{surf}}{T}$. Such a behavior of the surface free energy can be derived within the exactly solvable model of surface deformations known as Hills and Dales Model [18]. Moreover, very recently using the relation between the tension of confining color string and the surface tension of the QGP bag derived in [19] it was possible to demonstrate [19, 20] that at the cross-over region the surface tension coefficient of large bags is unavoidably negative. In addition this approach allows one to determine such an important parameter of the QGBSTM as the value of triCEP/CEP temperature $T_{cep} = 153.9 \pm 4.5$ MeV [21] under the plausible assumptions which are typical for the liquid-gas PT [13, 18]. Note that this value of the triCEP/CEP temperature is in a good agreement with the lattice QCD results discussed in [2].

In the vicinity of triCEP the behavior of both the deconfinement PT curve and the nil surface tension coefficient line in the $\mu - T$ plane is parameterized via a single parameter $\xi^T > 0$:

$$T_{cep} - T_\Sigma(\mu) \sim T_{cep} - T_c(\mu) \sim (\mu - \mu_{cep})^{\xi^T}\,, \tag{6}$$

since, as discussed above, $T_c(\mu)$ and $T_\Sigma(\mu)$ are tangent to each other at triCEP. This is a new index which was not considered both in the FDM and SMM. As it will be shown below this index is responsible for a new universality class compared to other exactly solvable models.

At the phase coexistence curve the Clapeyron-Clausius equation can be written as $\frac{d\mu_c}{dT} = -\frac{S_H - S_Q}{\rho_H - \rho_Q}\Big|_{T=T_c}$. The entropy density S_A and the baryonic density ρ_A are, respectively, defined as T and μ partial derivatives of the corresponding pressure p_A, where $A \in \{H, Q\}$. Then using (1) and (2) the Clapeyron-Clausius equation can be explicitly rewritten as

$$\frac{d\mu_c}{dT} = -\frac{\frac{\partial F_H}{\partial T} + \frac{\partial u}{\partial T} I_\tau(0,\Sigma) + \frac{\partial s_Q}{\partial T}\left(\frac{\partial F_H}{\partial s} - 1\right) - \frac{\partial \Sigma}{\partial T} u I_{\tau-\varkappa}(0,\Sigma)}{\frac{\partial F_H}{\partial \mu} + \frac{\partial u}{\partial \mu} I_\tau(0,\Sigma) + \frac{\partial s_Q}{\partial \mu}\left(\frac{\partial F_H}{\partial s} - 1\right) - \frac{\partial \Sigma}{\partial \mu} u I_{\tau-\varkappa}(0,\Sigma)}\Bigg|_{T=T_c}. \tag{7}$$

Let us parameterize the behavior of the numerator and denominator in (7) at the triCEP vicinity as

$$\left[\frac{\partial F_H}{\partial T} + \frac{\partial u}{\partial T} I_\tau(0,\Sigma) + \frac{\partial s_Q}{\partial T}\left(\frac{\partial F_H}{\partial s} - 1\right)\right]_{T=T_c} \sim \ (T_{cep} - T_c(\mu))^{\chi + \frac{1}{\xi^T} - 1}, \quad (8)$$

$$\left[\frac{\partial F_H}{\partial \mu} + \frac{\partial\, u}{\partial \mu} I_\tau(0,\Sigma) + \frac{\partial s_Q}{\partial \mu}\left(\frac{\partial F_H}{\partial s} - 1\right)\right]_{T=T_c} \sim \ (T_{cep} - T_c(\mu))^{\chi}, \quad (9)$$

where $\chi \geq \max(0, 1 - \frac{1}{\xi^T})$ denotes another new index. The latter inequality follows from the fact that the integral I_τ and functions F_H, u, s_Q together with their derivatives are finite for any finite values of T and μ. It is necessary to emphasize that the index χ unavoidably appears from an inspection of the Clapeyron-Clausius equation. Since the latter is a direct consequence of the Gibbs criterion for phase equilibrium, then an introduction of the parameter χ is quite general and could be done for any model with the PT of the liquid-gas type. So far, the index χ was never used for the calculation of critical exponents. For example, in the SMM case this parameter is set to zero by construction [22, 23].

III. THE CRITICAL INDICES OF THE QGBSTM

The standard set of critical exponents α', β and γ [6, 24] describes the T-dependence of the system near triCEP:

$$C_\rho \ \sim \ |t|^{-\alpha'}, \ \text{for} \quad t \leq 0 \quad \text{and} \quad \rho = \rho_{cep}, \quad (10)$$

$$\Delta\rho \ \sim \ |t|^{\beta}, \quad \text{for} \quad t \leq 0, \quad (11)$$

$$\Delta K_T \ \sim \ |t|^{-\gamma'}, \ \text{for} \quad t < 0, \quad (12)$$

where $\Delta\rho \equiv (\rho_Q - \rho_H)_{T=T_c}$ defines the order parameter, $C_\rho \equiv \frac{T}{\rho}(\frac{\partial S}{\partial T})_\rho$ denotes the specific heat at the critical density and $\Delta K_T \equiv (K_T^H - K_T^Q)_{T=T_c}$ is the discontinuity in the isothermal compressibility $K_T \equiv \frac{1}{\rho}(\frac{\partial \rho}{\partial p})_T$ across the PT line, the variable t is the reduced temperature $t \equiv \frac{T - T_{cep}}{T_{cep}}$. The critical isotherm shape is given by the index δ [6, 24]

$$p_{cep} - \widetilde{p} \sim (\rho_{cep} - \widetilde{\rho})^{\delta} \ \text{for} \quad t = 0. \quad (13)$$

Hereafter the tilde indicates that $T = T_{cep}$.

The calculation of α' requires the knowledge of the specific heat behavior $C_\rho \equiv \frac{T}{\rho}(\frac{\partial S}{\partial T})_\rho$ along the critical isochore $\rho = \rho_{cep}$ inside the mixed quark-gluon-hadron phase. In contrast to the works [22, 24], the famous Yang-Yang formula [25] is not used here to calculate C_ρ, since this formula leads to a less convenient representation. Therefore, below there are given some details of the specific heat evaluation.

As usual the entropy density and baryonic density of the mixed phase are defined via that ones of the pure phases and the parameter λ, which is the volume fraction of hadronic phase (i.e. the volume fraction of QGP is, respectively, $1 - \lambda$):

$$\rho|_{T=T_c} \ = \ \lambda \rho_H|_{T=T_c} + (1-\lambda)\rho_Q|_{T=T_c}, \quad (14)$$

$$S|_{T=T_c} \ = \ \lambda S_H|_{T=T_c} + (1-\lambda) S_Q|_{T=T_c}. \quad (15)$$

Varying λ from 0 to 1 one can describe any state in the mixed phase for fixed μ or T, since these variables are not independent inside the mixed phase. Replacing $\rho|_{T=T_c}$ by ρ_{cep} in Eq. (14), one can calculate the total T-derivative of λ along the critical isochore $\rho = \rho_{cep}$. Then for $\rho = \rho_{cep}$ from Eq. (15) one finds

$$C_\rho = \frac{T}{\rho_{cep}}\left[\lambda\left(\frac{d}{dT}(S_H - S_Q)_{T=T_c} + \frac{d\mu_c}{dT}\frac{d}{dT}(\rho_H - \rho_Q)_{T=T_c}\right) + \left(\frac{d(S_Q|_{T=T_c})}{dT} + \frac{d\mu_c}{dT}\frac{d(\rho_Q|_{T=T_c})}{dT}\right)\right]. \quad (16)$$

Using the Clapeyron-Clausius equation it is possible to rewrite the specific heat (16) in the form

$$C_\rho = \frac{T}{\rho_{cep}}\left[(\rho_Q - \rho_{cep})_{T=T_c}\frac{d^2\mu_c}{dT^2} + \left(\frac{d(S_Q|_{T=T_c})}{dT} + \frac{d\mu_c}{dT}\frac{d(\rho_Q|_{T=T_c})}{dT}\right)\right], \quad (17)$$

where the definition (14) for λ was used.

The critical exponent α' describes the temperature behavior of the most singular term in Eq. (17). Expanding $\rho_Q|_{T=T_c}$ into series of t-powers and using the parametrization (6) of the coexistence curve one can show that the first

term in (17) behaves as $t^{\min(1,\frac{1}{\xi^T})+\frac{1}{\xi^T}-2}$. Since the entropy density and baryonic density of QGP are defined via the T and μ derivatives of the function $Ts_Q(T,\mu)$, which is a regular function together with its first and second derivatives, then the singularity in the second term of (17) appears from $\left(\frac{d\mu_c}{dT}\right)^2 \sim t^{\frac{2}{\xi^T}-2}$. Clearly, the second term in (17) is a non-vanishing constant, if T-derivative of $\mu_c(T)$ is finite at triCEP. Accounting for this fact, one gets the critical exponent α' as

$$\alpha' = 2 - 2\min\left(1, \frac{1}{\xi^T}\right). \tag{18}$$

This equation shows that $\alpha' > 0$ for $\xi^T > 1$ only, otherwise $\alpha' = 0$. As it will be seen below only the index $\alpha' > 0$ depends on ξ^T, whereas other critical exponents always have the branch of solutions which is independent on ξ^T. Compared to the FDM and SMM, such a property leads to a principally new and unique possibility to choose the index α' independently of other critical indices.

To calculate the critical exponents β and γ' it is necessary to analyze the behavior of the integral $I_{\tau-q}(0,\Sigma)$ for small positive values of Σ, which is the case, when $T = T_c(\mu)$ approaches T_{cep}. In the limit $\Sigma \to +0$ the integral $I_{\tau-q}(0,\Sigma)$ remains finite for $\tau > 1+q$ and diverges otherwise. For $\tau = 1+q$ it diverges logarithmically. Indeed, the substitution $z \equiv v\,\Sigma^{\frac{1}{\varkappa}}$ yields [22, 23]

$$I_{\tau-q}(0,\Sigma) = \Sigma^{\frac{\tau-1-q}{\varkappa}} \int\limits_{V_0\Sigma^{\frac{1}{\varkappa}}}^{\infty} dz\, \frac{e^{-z^{\varkappa}}}{z^{\tau-q}}\,. \tag{19}$$

The condition $\tau < q+1$ guaranties a convergence of the integral (19) at its lower limit. The finite values of $I_{\tau-q}(0,\Sigma)$ at the upper limit is provided by an exponential factor. Summarizing these results for $\Sigma \to +0$ one can write

$$I_{\tau-q}(0,\Sigma) \sim \begin{cases} \Sigma^{\frac{\tau-1-q}{\varkappa}}, & \text{for} \quad \tau < 1+q \\ \ln\Sigma, & \text{for} \quad \tau = 1+q \\ \text{const}, & \text{for} \quad \tau > 1+q \end{cases} \sim\ \Sigma^{\min\left(0,\frac{\tau-1-q}{\varkappa}\right)}, \tag{20}$$

where the limit $(x^y \ln x)_{x\to 0} \sim x^y$ was used.

To calculate the index δ one should investigate the integral (4) in the limit $\widetilde{\Delta} s \to +0$ and $\widetilde{\Sigma} \to -0$, but for a condition $\frac{\widetilde{\Sigma}}{\widetilde{\Delta} s^{\varkappa}} \to -0$ which, as shown in [11], is the case (also see Eq. (28) for details). Then one obtains

$$I_{\tau-q}(\widetilde{\Delta} s,\widetilde{\Sigma}) \sim \begin{cases} \widetilde{\Delta} s^{\tau-1-q}, & \text{for} \quad \tau < 1+q \\ \ln\widetilde{\Delta} s, & \text{for} \quad \tau = 1+q \\ \text{const}, & \text{for} \quad \tau > 1+q \end{cases} \sim\ \widetilde{\Delta} s^{\min(0,\tau-1-q)}. \tag{21}$$

Using the definition of the baryonic density and Eqs. (1) and (2) one can find the baryonic density discontinuity across the deconfinement PT line:

$$\Delta\rho = T_c \cdot \left. \frac{\frac{\partial F_H}{\partial\mu} + \frac{\partial u}{\partial\mu} I_\tau(0,\Sigma) + \frac{\partial s_Q}{\partial\mu}\left(\frac{\partial F_H}{\partial s} - 1\right) - \frac{\partial\Sigma}{\partial\mu} u I_{\tau-\varkappa}(0,\Sigma)}{1 - \frac{\partial F_H}{\partial s} + u I_{\tau-1}(0,\Sigma)} \right|_{T=T_c}. \tag{22}$$

According to the parameterizations (5) and (6) one obtains that along the PT line $\Sigma \sim t^{\zeta}$ and $\frac{\partial\Sigma}{\partial\mu} \sim t^{\zeta-\frac{1}{\xi^T}}$ and, hence, the temperature dependence of $\Delta\rho$ in (22) can be straightforwardly found from Eqs. (9) and (20). This yields

$$\beta = \frac{\zeta}{\varkappa}(2-\tau) + \begin{cases} \chi, & \text{for} \quad \chi \le \frac{\zeta}{\varkappa}\min(\varkappa,\tau-1) - \frac{1}{\xi^T} \\ \frac{\zeta}{\varkappa}\min(\varkappa,\tau-1) - \frac{1}{\xi^T}, & \text{for} \quad \chi \ge \frac{\zeta}{\varkappa}\min(\varkappa,\tau-1) - \frac{1}{\xi^T} \end{cases}. \tag{23}$$

The structure of this equation is obvious: its first term corresponds to the denominator of Eq. (22), whereas the second one describes the leading term of the numerator of (22).

To find the critical exponent γ' one has to calculate the isothermal compressibility $K_T \equiv \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_T$ both for QGP and for hadronic phase. With the help of the baryonic density definition one can rewrite the isothermal compressibility as $K_T = \frac{1}{\rho^2}\frac{\partial^2 p}{\partial\mu^2}$. This result clearly demonstrates that the QGP contribution into the ΔK_T is negligibly small since the QGP pressure is given by the function $s_Q(T,\mu)$, which, according to the QGMSTM assumptions [11], is finite

together with its first and second derivatives for all finite values of T and μ. Therefore, at triCEP ΔK_T can diverge due to the hadronic phase contribution only, i.e. near triCEP $\Delta K_T \simeq K_T^H$. Therefore, using Eq. (2) and keeping at triCEP the most singular terms one finds

$$\Delta K_T \simeq \left[\frac{T}{\rho_H^2} \frac{\left(\frac{\partial \Delta s}{\partial \mu}\right)^2 u I_{\tau-2}(0,\Sigma)}{1 - \frac{\partial F_H}{\partial s} + u I_{\tau-1}(0,\Sigma)} \right]_{T=T_c(\mu)} \sim \left. \frac{I_{\tau-2}(0,\Sigma)}{I_{\tau-1}(0,\Sigma)} \cdot \left(\frac{\partial \Delta s}{\partial \mu}\right)^2 \right|_{T=T_c(\mu)} . \tag{24}$$

From the definition of Δs it follows that along the PT line $\frac{\partial \Delta s}{\partial \mu} = \frac{\rho_H - \rho_Q}{T} \sim t^\beta$. Then from Eqs. (20), (5), (6) one has

$$\gamma' = \frac{\zeta}{\varkappa} - 2\beta. \tag{25}$$

At the critical isotherm one can use the definition of Δs to get

$$\widetilde{p} - p_{cep} = T_{cep}(\widetilde{\Delta} s + \widetilde{s}_Q - s_Q|_{cep}) = T_{cep} \left(\widetilde{\Delta} s + \Delta\mu \left. \frac{\partial s_Q}{\partial \mu} \right|_{cep} \right), \tag{26}$$

where in the second step one has to expand $\widetilde{s}_Q$ in powers of $\Delta\mu \equiv \mu - \mu_{cep}$ and keep the linear term. Similarly one determines the deviation of the baryonic density taken at the critical isotherm that is lying outside the mixed phase from the baryonic density at triCEP:

$$\widetilde{\rho} - \rho_{cep} = T_{cep} \left(\frac{\partial \widetilde{\Delta} s}{\partial \mu} + \frac{\partial \widetilde{s}_Q}{\partial \mu} - \left. \frac{\partial s_Q}{\partial \mu} \right|_{cep} \right) = T_{cep} \left(\frac{\partial \widetilde{\Delta} s}{\partial \mu} + \Delta\mu \left. \frac{\partial^2 s_Q}{\partial \mu^2} \right|_{cep} \right). \tag{27}$$

Now it is clear that the behavior of $\widetilde{\Delta} s$ near triCEP should be analyzed in order to calculate the critical exponent δ. Let us consider the case $\chi = 0$ first. Such a case is typical for the SMM [15, 22]. Since at triCEP $\widetilde{\Delta} s = 0$, then substituting the expansion $I_\tau(\widetilde{\Delta} s, \widetilde{\Sigma}) \approx I_\tau(0,0) - \widetilde{\Delta} s I_{\tau-1}(\frac{\widetilde{\Delta} s}{2}, \frac{\widetilde{\Sigma}}{2}) - \widetilde{\Sigma} I_{\tau-\varkappa}(\frac{\widetilde{\Delta} s}{2}, \frac{\widetilde{\Sigma}}{2})$ [11] into Eq. (2) for the cross-over states one gets

$$\widetilde{\Delta} s = \frac{\Delta\mu \left(\frac{\partial F_H}{\partial \mu} + \frac{\partial u}{\partial \mu} I_\tau(0,0) + \frac{\partial s_Q}{\partial \mu} \left(\frac{\partial F_H}{\partial s} - 1 \right) \right)_{cep} - \widetilde{\Sigma}\, \widetilde{u}\, I_{\tau-\varkappa}(\frac{\widetilde{\Delta} s}{2}, \frac{\widetilde{\Sigma}}{2})}{1 - \frac{\partial \widetilde{F}_H}{\partial \mu} + \widetilde{u}\, I_{\tau-1}(\frac{\widetilde{\Delta} s}{2}, \frac{\widetilde{\Sigma}}{2})}, \tag{28}$$

where it is sufficient to keep only the first order terms of expansion, whereas for the case $\chi > 0$ one has to keep the second order terms as well. With the help of Eq. (21) one concludes that

$$\widetilde{\Delta} s \sim \begin{cases} \Delta\mu^{\frac{\xi^T \zeta}{\max(\tau-1,\varkappa)}}, & \text{for } \frac{\xi^T \zeta}{\max(\tau-1,\varkappa)} \leq \frac{1}{\tau-1} \\ \Delta\mu^{\frac{1}{\tau-1}}, & \text{for } \frac{\xi^T \zeta}{\max(\tau-1,\varkappa)} \geq \frac{1}{\tau-1} \end{cases}, \tag{29}$$

which allows one to find the index δ for $\chi = 0$

$$\delta|_{\chi=0} = \begin{cases} \left[\frac{\xi^T \zeta}{\max(\tau-1,\varkappa)} - 1 \right]^{-1}, & \text{for } 1 < \frac{\xi^T \zeta}{\max(\tau-1,\varkappa)} \leq \frac{1}{\tau-1} \\ \frac{\tau-1}{2-\tau}, & \text{for } \frac{\xi^T \zeta}{\max(\tau-1,\varkappa)} \geq \frac{1}{\tau-1} \end{cases}. \tag{30}$$

Note that in this case the inequalities $\frac{3}{2} < \tau \leq 2$ providing the existence of the 2-nd order PT at triCEP [11] also guaranty the fulfillment of the condition $\delta > 1$.

The result for the index δ in the case $\chi > 0$ can be found similarly:

$$\delta|_{\chi>0} = \left[\frac{\xi^T \zeta}{\max(\tau-1,\varkappa)} - 1 \right]^{-1}, \quad \text{for} \quad 1 < \frac{\xi^T \zeta}{\max(\tau-1,\varkappa)} \leq \frac{2}{\tau-1}. \tag{31}$$

Since in some aspects the QGBSTM is similar to the FDM and SMM it is interesting to compare its critical exponents with that ones of the FDM [13] and SMM [22]. However, one has also to remember that in contrast to the FDM, both the QGBSTM and the SMM have a non-vanishing excluded volume of the constituents and the same range of the Fisher exponent $1 < \tau \leq 2$ for the triCEP existence, whereas the FDM can be formulated for $\tau > 2$

only. Since the FDM and SMM implicitly treat the parameter $\chi = 0$, then it is most natural to compare their critical exponents with the QGBSTM results just for such a case. Eq. (30) shows that in the QGBSTM there is a regime, when its index $\delta|_{\chi=0}$ matches the SMM result [22]. Moreover, it is easy to see that in this regime all other indices the QGBSTM and SMM coincide for $\xi^T \leq 1$

$$\alpha' = 0\,, \quad \beta = \frac{\zeta}{\varkappa}(2-\tau)\,, \quad \gamma' = \frac{2\,\zeta}{\varkappa}\left(\tau - \frac{3}{2}\right) \quad \text{and} \quad \delta|_{\chi=0} = \frac{\tau-1}{2-\tau}\,, \tag{32}$$

although the present model has entirely new regime for $\xi^T > 1$, for which all its indices except for α' are equal to the corresponding exponents of the SMM, whereas α' can be chosen freely provided that the index ξ^T exceeds the value $\frac{\max(\tau-1,\varkappa)}{\zeta\,(\tau-1)}$. On the other hand the QGBSTM always belongs to a different universality class $1 < \tau \leq 2$ (except for a special case $\tau = 2$), than that one of FDM in which $\tau \geq 2$. Thus, the spectrum of values of the QGBSTM critical indices is more rich than the corresponding spectra of the FDM and SMM since this model contains two new indices ξ^T and χ.

Let us demonstrate this using a few sets of critical indices. Since the present model contains five parameters, τ, $\varkappa$, ζ, ξ^T and χ, there exists an infinite number of possibilities to describe the four standard critical exponents, α', β, γ' and δ. Thus, the critical exponents of the 2-dimensional Ising model [29] shown in the first row of the Table I can be exactly reproduced by many sets of parameters with the vanishing value of index χ (see the first row in the Table II). It is interesting to mention that, if one uses the linear T-dependence of the surface tension coefficient (5) employed in the FDM for $T \leq T_{cep}$, i.e. fixes $\zeta = 1$, then for the 2-dimensional Ising model one obtains $\varkappa = \frac{1}{2}$, which is typical for the dimension $d = 2$. However, in contrast to the FDM, where $\tau = \frac{31}{15} > 2$, this set of critical exponents is described by the value $\tau = \frac{31}{16} < 2$.

It is interesting to analyze the critical indices of the simple liquids since near the deconfinement region the QGP behaves as a strongly interacting liquid [1]. Taking the wide range of values for the critical exponents of simple liquids [29] (see the Table I) one can describe them in many ways. The set A in the Table II demonstrates one of such possibilities, but its value of the parameter $\varkappa = (\tau-1)\frac{437}{720} \leq \frac{437}{720} \approx 0.6069$ is essentially smaller than the expected for 3-dimensions value $\varkappa \approx \frac{2}{3}$. At the first glance it seems that the accuracy of above 10% for the set A value of $\varkappa$ which is obtained for $\tau < 2$ is acceptable, but a careful study of the surface free energy in various cluster models [13, 24, 30] shows that for 3-dimensional case the value $\varkappa = \frac{2}{3}$ holds with much better accuracy of about 4%. However, if one requires that the critical exponents of simple liquids are reproduced for $\varkappa > 0.49$, i.e. in such a way that both 2- and 3-dimensional values of parameter $\varkappa$ can be included, then it can be shown that the only existing solution corresponds to the SMM values for indices β, γ' and δ (32) with $\chi = 0$, while the index α' is fixed to its experimental value (see the set B for liquids in the Table II). In fact, the same outcome is obtained for the critical exponents of the 3-dimensional Ising model (see the Table II) which were found with very high accuracy in [31] and are given in the Table I.

There are two important consequences that follow from these results. First, since the QGBSTM indices β, γ' and $\delta|_{\chi=0}$ match that ones of the SMM and the latter do not depend on the parameter ξ^T, then there exist two relations for index τ [22]

$$\tau = 2 - \frac{1}{1+\delta} \quad \text{and} \quad \tau = 2 - \frac{\beta}{\gamma' + 2\beta}\,. \tag{33}$$

The first of these equalities follows from the definition of index $\delta|_{\chi=0} = \frac{\tau-1}{2-\tau}$, whereas the second one can be obtained directly from Eq. (32). Second, from (33) it follows that the index $\tau \approx 1.826 \pm 0.02$ has a very narrow range for the set B of simple liquids and for the 3-dimensionl Ising model (see the Table II). Both of these consequences are important for QCD since its universality class is expected to match the class of the 3-dimensionl Ising model [2, 32, 33]. Therefore, on the basis of above results one can expect [21] that at triCEP the volume distribution of large QGP bags has a power like form $v^{-\tau}$ with $\tau \approx 1.826 \pm 0.02$. The same conclusion for the mass distribution results from the fact that in the QGBSTM [11] the mass of large bags and its volume $v > V_0$ are proportional to each other.

IV. THE SCALING RELATIONS OF THE QGBSTM

The well known exponent inequalities were proven for real gases by

$$\text{Fisher}\,[26]: \qquad \alpha' + 2\beta + \gamma' \geq 2, \tag{34}$$

$$\text{Griffiths}\,[27]: \qquad \alpha' + \beta(1+\delta) \geq 2, \tag{35}$$

$$\text{Liberman}\,[28]: \qquad \gamma' + \beta(1-\delta) \geq 0. \tag{36}$$

	α'	β	γ'	δ
2d Ising model	0	$\frac{1}{8}$	$\frac{7}{4}$	15
Simple liquids	0.09-0.11	0.32-0.35	1.2-1.3	4.2-4.8
3d Ising model	0.1096 ± 0.0005	0.3265 ± 0.0001	1.2373 ± 0.0002	4.7893 ± 0.0008

TABLE I: The critical indices of simple liquids [29], 2-dimensional Ising model [29] and 3-dimensional Ising model [31].

	χ	ξ^T	τ	$\varkappa$	ζ
2d Ising model	0	$\frac{8}{15} \le \xi^T \le 1$	$\frac{31}{16}$	$\min(2\varkappa, \frac{15}{8}) \ge \frac{1}{\xi^T}$	$2\varkappa$
Simple liquids A	$0 < \chi < \frac{8}{23}$	$\frac{20}{19}$	$\tau = \frac{20}{11} + \chi\frac{23}{44}$	$(\tau - 1)\frac{437}{720}$	$\frac{44}{23}\varkappa$
Simple liquids B	0	1.0526 ± 0.0055	1.8255 ± 0.0212	$\varkappa \ge 0.4947$	$\varkappa \cdot (1.92 \pm 0.026)$
3d Ising model	0	1.0579 ± 0.000055	1.8272 ± 0.000048	$\varkappa \ge 0.4999$	$\varkappa \cdot (1.8903 \pm 0.00007)$

TABLE II: The QGBSTM parameters that describes the corresponding exponents given in the Table I.

The corresponding exponent inequalities for magnetic systems are often called Rushbrooke, Griffiths and Widom inequalities, respectively. These inequalities are traditionally believed to play a fundamental role in the modern theory of critical phenomena. However, the real situation with the scaling inequalities (34)–(36) is not that trivial as it is often presented in the textbooks. Thus, as one can see from the Table III the inequalities (34)–(36) exactly hold for the Onsager solution only, whereas for simple liquids only the Fisher relation is obeyed and even for highly accurate numerical evaluation of the critical exponents of the 3-dimensional Ising model there are some problems with the inequalities (34) and (36). Moreover, a long time ago it was found [24] that the traditional definition of the exponent α' given by (10) may lead to somewhat smaller value than 2 staying on the right hand side of Eqs. (34) and (35). A similar result was analytically found for the SMM [22, 23], which shows that for the standard set of the SMM parameters [14, 15] the right hand side of inequalities (34) and (35) should be replaced by $\frac{15}{8}$. Therefore, it is interesting to verify the scaling inequalities for the QGBSTM indices obtained here.

Despite the usual expectations, the QGBSTM critical exponents do not obey the traditional scaling relations in general. Again, as in the SMM case, the Fisher and Griffiths inequalities are not always fulfilled, whereas the Liberman inequality is fulfilled for any values of the model parameters. Indeed, let's demonstrate the validity of the Liberman inequality (36) first. For simplicity, consider the case $\delta|_{\chi=0} = \frac{\tau-1}{2-\tau}$ of Eq. (30), which is realized for $\chi = 0$. As it was mentioned in the preceding section the QGBSTM indices β, γ' and $\delta|_{\chi=0}$ for this case coincide with the corresponding exponents of the SMM and, hence, as in the SMM case [22], the Liberman inequality is fulfilled within the present model for any choice of α'. This, however, can be shown from the explicit expressions for the indices β, γ' and $\delta|_{\chi=0}$, i.e. from Eqs. (23), (25) and (30). Using these equations one obtains

$$\gamma' + \beta(1-\delta) = -\frac{\min\left(0, \frac{\zeta}{\varkappa}\min(\tau-1, \varkappa) - \frac{1}{\xi^T}\right)}{2-\tau} \ge 0, \tag{37}$$

where the validity of the right hand side of (37) easily follows now from the inequalities $\min(0, \ldots) \le 0$ and $\tau < 2$. The Liberman relation analysis for other values of the index δ gives the same result. Using the Liberman inequality and the explicit expressions for the QGBSTM critical exponents one can get the following result for the Fisher and Griffiths inequalities

$$\alpha' + \beta(\delta+1) \le \alpha' + 2\beta + \gamma' = 2 + \left[\frac{\zeta}{\varkappa} - 2\min\left(1, \frac{1}{\xi^T}\right)\right], \tag{38}$$

which holds for any value of the index χ. This equation clearly demonstrates that the Fisher and Griffiths inequalities are not obeyed for the values of parameters satisfying the inequality $\frac{\zeta}{\varkappa} < 2\min(1, \frac{1}{\xi^T})$. Moreover, a fulfillment of the Fisher scaling inequality does not guaranty that the Griffiths one is obeyed.

In order to 'save' the scaling inequalities (34) and (35) it was suggested [24] to replace the index α' by α'_s, where the index α'_s describes the temperature dependence of the specific heat difference $\Delta C = (C_{\rho_H} - C_{\rho_Q})_{T=T_c}$ for two phases. Since at the coexistence curve the specific heat of each phase is defined by the total T derivative of the entropy, then,

	Fisher: $\alpha' + 2\beta + \gamma'$	Griffiths: $\alpha' + \beta(\delta + 1)$	Liberman: $\gamma' + \beta(1 - \delta)$
2D Ising model	2	2	0
Simple liquids	2.02 ± 0.0055	1.9425 ± 0.0055	0.0775 ± 0.0212
3D Ising model	1.99996 ± 0.00007	2.000412 ± 0.005	-0.000052 ± 0.002

TABLE III: Scaling relations between the critical exponents taken from the Table I. The uncertainties were calculated from their values given in the Table I using the error determination method for indirect measurements [34].

using the Clapeyron-Clausius equation one can find ΔC as

$$\Delta C = \frac{T_c}{\rho_H} \frac{d}{dT} \left[T_c \frac{d\mu_c}{dT} (\rho_Q - \rho_H) \right] + T_c \frac{\rho_Q - \rho_H}{\rho_H \rho_Q} \frac{dS_Q}{dT}, \tag{39}$$

where the entropy density S_Q and the baryonic densities ρ_Q, ρ_H are calculated along the deconfinement PT line. Then from the parameterization (6) and the definition of index β one gets

$$\alpha'_s = \begin{cases} 2 - \beta - \frac{1}{\xi^T}, & \text{for} \quad \frac{1}{\xi^T} < 2 \\ -\beta, & \text{for} \quad \frac{1}{\xi^T} \geq 2 \end{cases} = \max\left(2, \frac{1}{\xi^T}\right) - \beta - \frac{1}{\xi^T}. \tag{40}$$

Note that $\alpha'_s \geq 0$ for $\frac{1}{\xi^T} < 2 - \beta$ only. Using α'_s from (40) instead of α' in the Fisher inequality, one can write

$$\alpha'_s + 2\beta + \gamma' = \max\left(2, \frac{1}{\xi^T}\right) + \left(\frac{\zeta}{\varkappa} - \beta - \frac{1}{\xi^T}\right) \geq 2 - \frac{\zeta}{\varkappa} \min(0, 1 + \varkappa - \tau) \geq 2, \tag{41}$$

where the final result follows from the obvious inequalities $\max(2, ...) \geq 2$ and $\min(0, ...) \leq 0$. Similarly, one can prove the validity of the Griffiths inequality for the index α'_s. Thus, the Fisher hypothesis [24] to replace α' by α'_s recovers the scaling inequalities for the critical exponents, however, it does not seem that such a suggestion is the final solution of this problem.

V. CONCLUSIONS

The critical indices of the QGBSTM with the triCEP are determined and compared with the critical exponents of other models. The QGBSTM critical exponents are expressed in terms of the model parameters τ, $\varkappa$, ζ and two newly introduced indices $\xi^T > 0$ and $\chi \geq \max(0, 1 - \frac{1}{\xi^T})$. The index ξ^T in (6) characterizes the behavior of the PT curve in the vicinity of triCEP in the plane of baryonic chemical potential and temperature, whereas the index χ in (9) describes the temperature dependence of a certain combination of the μ-derivatives of the model spectrum in the same vicinity.

Since in the FDM and SMM the index χ is implicitly set to zero, while the index ξ^T either does not exists (FDM) or is fixed by other model parameters (SMM), then the spectrum of the values of critical exponents of the present model in more rich compared to those models. For the case $\chi = 0$ and $\xi^T \leq 0$ the QGBSTM reproduces the critical exponents of the SMM with triCEP, whereas for other choice of parameters χ and ξ^T these models belong to different classes of universality. Also the universality classes of the FDM and QGBSTM are different since the range of values of their index τ is different (except for a singular case $\tau = 2$): $\tau \geq 2$ in the FDM and $1 < \tau \leq 2$ in the QGBSTM. A very important result of the present work is that, if one requires that the parameter $\varkappa$ describing the surface dependence on the bag volume should have the values typical for dimensions 2 and 3, then the critical indices of simple liquids and 3-dimensional Ising model can be described only by the SMM expressions for β, γ' and δ exponents while the index α' differs from the SMM value. For these sets of critical exponents a very narrow range of parameter $\tau = 1.826 \pm 0.02$ is found. Such a prediction might be important for experimental searches of the QCD phase diagram endpoint since just this exponent describes the power law in the volume distribution of large bags at triCEP.

The direct calculations show that for the standard definition of the critical index α' (found along the critical isochore) the Fisher and Griffiths scaling inequalities are not always fulfilled, whereas the Liberman inequality is obeyed for any values of the model parameters. In contrast to the SMM, in which the critical isochore belongs to the boundary of the mixed and liquid phases, the critical isochore of the present model is located inside the mixed phase and, hence, the conditions of the Fisher theorem [26] proving the validity of (34) are formally satisfied, but

the Fisher and Griffiths scaling inequalities are not fulfilled. Therefore, it is quite possible that instead of the Fisher suggestion one should search for an alternative solution of this problem and, thus, to admit an existence of other, non-Fisher, universality classes of critical exponents for which the right hand side of scaling inequalities should be modified. Hopefully, further theoretical and experimental studies of this problem will find its final solution.

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